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Perfect 3-colorings of some generalized Peterson graph

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Abstract

The notion of a perfect coloring, introduced by Delsarte, generalizes the concept of completely regular code. A perfect z-colorings of a graph is a partition of its vertex set. It splits vertices into z parts $P_1, \cdots, P_z$ such that for all $i, j \in \{1, \cdots, z\}$, each vertex of P_i is adjacent to p_{ij}, vertices of P_j. The matrix $P = (p_{ij})_{i,j\in\{1,\cdots,z\}}$, is called parameter matrix. In this article, we classify all the realizable parameter matrices of perfect 3-colorings of some the generalized peterson graph.

Keywords: Parameter matrices, Perfect coloring, Equitable partition, Generalized peterson graph.

2020 MSC: 03E02, 05C15, 68R05.

1. Introduction

The concept of a perfect z-coloring plays a significant role in graph theory, algebraic combinatorics, and coding theory (completely regular codes). There is another phrase for this concept in the writing as "equitable partition" [9]. In 1973, Delsarte conjectured the non-existence of nontrivial perfect codes in Johnson graphs. Since then, some effort has been made to count the parameter matrices of some Johnson graphs, including $J(4,2)$, $J(5,2)$, $J(6,2)$, $J(6,3)$, $J(7,3)$, $J(8,3)$, $J(8,4)$, and $J(v,3)$ (v odd) [3, 4, 8].

Fon-Der-Flass count the parameter matrices (perfect 2-colorings) of n-dimensional hypercube Q_n for $n < 24$. He also obtained some constructions and a necessary condition for the existence of perfect 2-colorings of the n-dimensional cube with a given parameter matrix [5, 6, 7]. In this article, we classify the parameter matrices of all perefect 3-colorings of some generalized peterson graph.

Some generalized peterson graph including $GP(7,1)$, $GP(8,1)$, $GP(8,2)$ and $GP(8,3)$ given as follow:

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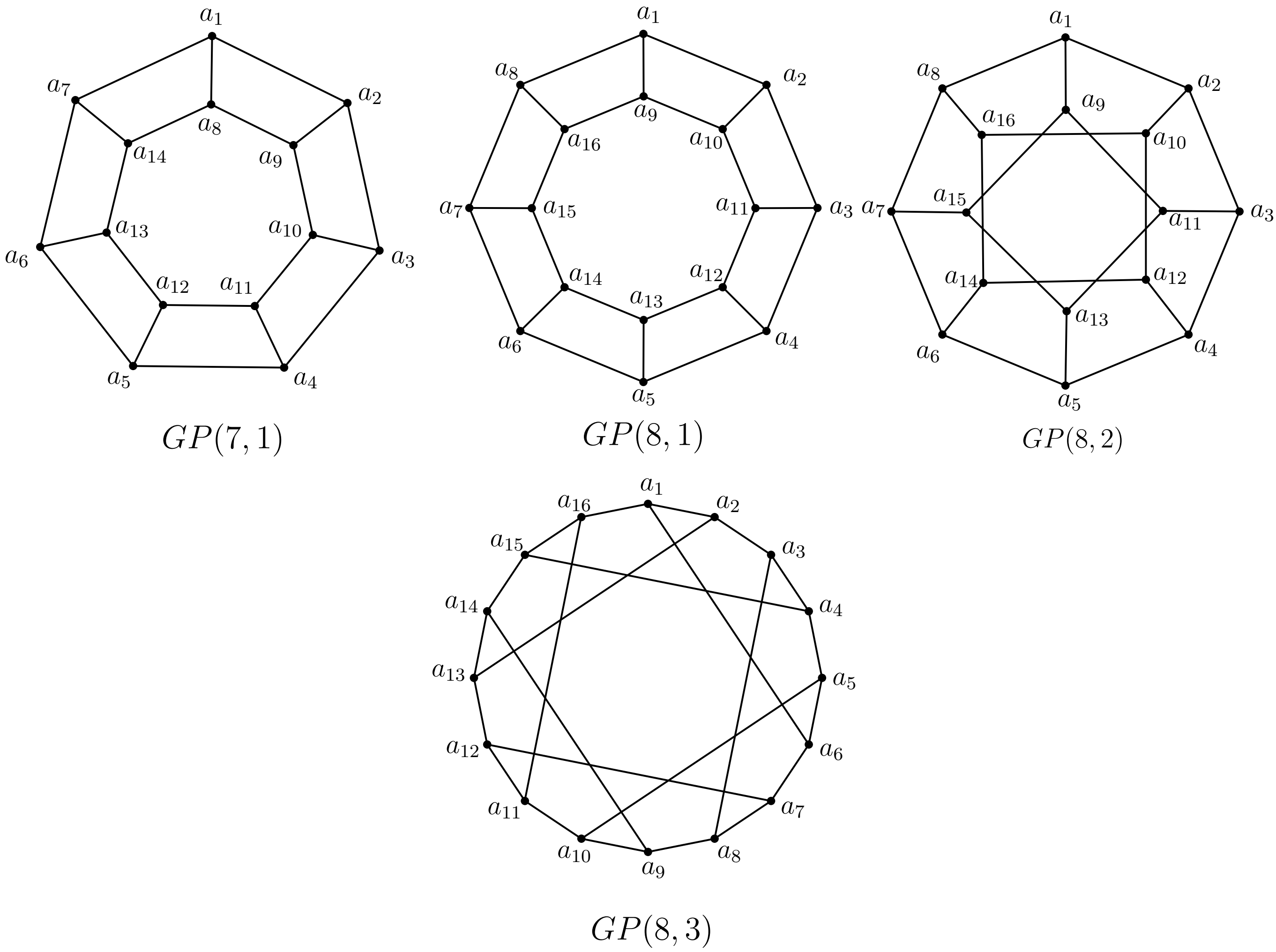

Figure 1: Some generalized peterson graph

Definition 1.1. The generalized peterson graph $\mathsf{GP}(n,k)$ has vertices, respectively, edges given by

$$V\big(\mathsf{GP}(n,k)\big) = \{a_i, b_i : 0 \leqslant i \leqslant n-1\},$$
$$E\big(\mathsf{GP}(n,k)\big) = \{a_i a_{i+1}, a_i b_i, b_i b_{i+k} : 0 \leqslant i \leqslant n-1\},$$

Where the subscripts are expressed as integers modulo n ($\geqslant 5$) , and k ($\geqslant 1$) is the skip.

Definition 1.2. For a graph G and an integer z, a mapping $T : V(G) \longrightarrow \{1, 2, \cdots, z\}$ is called a perfect z-coloring with matrix $P = (p_{ij})_{i,j \in \{1,\cdots,z\}}$, if it is surjective, and for all i, j, for every vertex of color i, the number of its neighbours of color j is equal to p_{ij} . The matrix P is called the parameter matrix of a perfect coloring. In the case $z = 3$, we call the first color white that show by W, the second color black that show by B and the third color red that show by R. In this article, we generally show a parameter matrix by

$$P = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Remark 1.3. In this paper, we consider all perfect 3-colorings, up to renaming the colors; i.e. We identify the perfect 3-coloring with the matrices

$$\begin{bmatrix} d & c & b \\ g & i & h \\ d & e & f \end{bmatrix}, \quad \begin{bmatrix} e & d & f \\ b & a & c \\ h & g & i \end{bmatrix}, \quad \begin{bmatrix} e & f & d \\ h & i & g \\ b & c & a \end{bmatrix}, \quad \begin{bmatrix} i & h & g \\ f & e & d \\ c & b & a \end{bmatrix}, \quad \begin{bmatrix} i & g & h \\ c & a & b \\ f & d & e \end{bmatrix}.$$

Obtained by switching the colors with original coloring .

2. Preliminaries

In this section, we present some results concerning necessary conditions for the existence of perfect 3-colorings of the generalized peterson graph of $GP(7,1)$, $GP(8,1)$, $GP(8,2)$ and $GP(8,3)$ with a given parameter matrix

$$P = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The simplest necessary condition for the existence of perfect 3-colorings of the generalized peterson

$$a + b + c = d + e + f = g + h + i = 3.$$

By using this condition and some computation, it is clear that we should consider 18 matrices .These matrices are listed below:

$$P_1 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix},$$

$$P_5 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad P_6 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}, \quad P_7 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \quad P_8 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix},$$

$$P_9 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \quad P_{10} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad P_{11} = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}, \quad P_{12} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix},$$

$$P_{13} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \quad P_{14} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \quad P_{15} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad P_{16} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$P_{17} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \quad P_{18} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

Theorem 2.1. *[9] If T is a perfect coloring of a graph G in z colors, then any eigenvalue of T is an eigenvalue of G.*

Theorem 2.2. *[1] Suppose that T is a perfect 3- coloring with matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, in the connected graph G.Then in this case, none of the following situations will occer.*

(1) $b = c = 0$,

(2) $d = f = 0$,

(3) $g = h = 0$,

(4) $b = 0 \leftrightarrow d = 0, c = 0 \leftrightarrow g = 0, h = 0 \leftrightarrow f = 0$.

Theorem 2.3. *[2] Let* T *a perfect 3-coloring of a graph* G *with matrix* $P = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

(1) If $b, c, f \neq 0$, *then*

$$|W| = \frac{|V(G)|}{\frac{b}{d} + 1 + \frac{c}{g}}, \qquad |B| = \frac{|V(G)|}{\frac{d}{b} + 1 + \frac{f}{h}}, \qquad |R| = \frac{|V(G)|}{\frac{h}{f} + 1 + \frac{g}{c}}.$$

(2) If $b = 0$, *then*

$$|W| = \frac{|V(G)|}{\frac{c}{g} + 1 + \frac{ch}{fg}}, \qquad |B| = \frac{|V(G)|}{\frac{f}{h} + 1 + \frac{fg}{ch}}, \qquad |R| = \frac{|V(G)|}{\frac{h}{f} + 1 + \frac{g}{c}}.$$

(3) If $c = 0$, *then*

$$|W| = \frac{|V(G)|}{\frac{b}{d} + 1 + \frac{bf}{dh}}, \qquad |B| = \frac{|V(G)|}{\frac{d}{b} + 1 + \frac{f}{h}}, \qquad |R| = \frac{|V(G)|}{\frac{h}{f} + 1 + \frac{dh}{bf}}.$$

(4) If $f = 0$, *then*

$$|W| = \frac{|V(G)|}{\frac{b}{d} + 1 + \frac{c}{g}}, \qquad |B| = \frac{|V(G)|}{\frac{d}{b} + 1 + \frac{cd}{bg}}, \qquad |R| = \frac{|V(G)|}{\frac{g}{c} + 1 + \frac{bg}{cd}}.$$

Theorem 2.4. *[1] If* $P = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ *be a parameter matrix of a* k*-regular graph, then the eigenvalues of* P *are*

$$\lambda_{1,2} = \frac{\operatorname{tr}(P) - k}{2} \pm \sqrt{\left(\frac{\operatorname{tr}(P) - k}{2}\right)^2 - \frac{\det(P)}{k}}, \qquad \lambda_3 = k.$$

Remark 2.5. The distinct eigenvalues of the graph $GP(7,1)$ are the numbers 3, 1, The distinct eigenvalues of graph $GP(8,1)$ are the numbers 3, 1, -1, The distinct eigenvalues of graph $GP(8,2)$ are the numbers 1, 3 and the distinct eigenvalues of graph $GP(8,3)$ are the numbers 3, 1, -1.

3. Perfect 3- colorings of some generalized peterson graph

The parameter matrices of $GP(7,1)$, $GP(8,1)$, $GP(8,2)$ and $GP(8,3)$ graphs are enumerated in the next teorems.

Theorem 3.1. *The graph* $GP(7,1)$ *has no perfect 3-colorings.*

Proof. A parameter matrix corresponding to perfect 3-colorings of the graph $GP(7,1)$ may be one of the matrices $P_1, \cdots, P_{18}$. By using Theorem 2.1 and Theorem 2.4, we can see that only the matrices $P_3, P_4, P_5, P_6, P_{10}, P_{12}, P_{15}$, and P_{18} can be a parameter matrices. By using Theorem 2.3, matrices P_3, P_6, P_{10}, P_{12}, and P_{15} cannot be a parameter matrices, because the number of white, black and red, are not an integer. For matrix P_4, each vertex with color white has three adjacent vertices with color red. Now have the following possibilities:

(1) $T(a_1) = T(a_{11}) = W$, $T(a_3) = T(a_5) = T(a_9) = T(a_{10}) = B$, $T(a_2) = T(a_4) = T(a_7) = T(a_8) = T(a_{12}) = R$ then $T(a_2) = T(a_{13}) = T(a_{14}) = B$, which is a contradiction with the second row of matrix P_4.

(2) $T(a_1) = T(a_8) = T(a_9) = T(a_{14}) = B$, $T(a_3) = W$, $T(a_2) = T(a_4) = T(a_6) = T(a_7) = T(a_{10}) = T(a_{13}) = R$ then $T(a_5) = T(a_{11}) = T(a_{12}) = B$, which is a contradiction with the second row of matrix P_4. Hence graph $GP(7, 1)$ has no perfect 3-colorings with matrix P_4.

Similar to matrix P_4, we proof for matrix P_5 and P_{18} as follows:

For matrix P_5, each vertex with color white has three adjacent vertices with color red. Now have the following possibilities:

(3) $T(a_3) = T(a_6) = W$, $T(a_7) = T(a_8) = T(a_{12}) = T(a_{14}) = B$, $T(a_2) = T(a_4) = T(a_5) = T(a_9) = T(a_{13}) = R$ then $T(a_4) = T(a_{10}) = R$, $T(a_{11}) = B$ which is a contradiction with the second row of matrix P_5.

(4) $T(a_1) = T(a_2) = T(a_5) = T(a_6) = T(a_{12}) = B$, $T(a_{10}) = W$, $T(a_3) = T(a_4) = T(a_8) = T(a_9) = T(a_{11}) = T(a_{13}) = R$ then $T(a_7) = B$ and $T(a_{14}) = W$, which is a contradiction with the second row of matrix P_5. Hence graph $GP(7, 1)$ has no perfect 3-colorings with matrix P_5.

For matrix P_{18}, each vertex with color white has two adjacent vertices with color black. Now have the following two possibilities:

(5) $T(a_1) = T(a_2) = T(a_3) = T(a_4) = T(a_5) = T(a_8) = T(a_{12}) = T(a_{14}) = R$, $T(a_7) = T(a_9) = T(a_{11}) = B$, $T(a_6) = W$ then $T(a_{13}) = R$, which is a contradiction with the second three of matrix P_{18}.

(6) $T(a_1) = T(a_4) = T(a_5) = T(a_9) = T(a_{10}) = T(a_{12}) = T(a_{14}) = R$, $T(a_2) = T(a_3) = B$, $T(a_8) = T(a_{11}) = W$ then $T(a_6) = T(a_7) = T(a_{13}) = B$, which is a contradiction with the second row of matrix P_{18}. Hence graph $GP(7, 1)$ has no perfect 3-colorings with matrix P_{18}.

□

Theorem 3.2. *The graph* $GP(8, 1)$ *has a perfect 3-colorings with the matrices* P_7 *and* P_{13}.

Proof. A parameter matrix corresponding to perfect 3-colorings of the graph $GP(8, 1)$ may be one of the matrices $P_1, \cdots, P_{18}$. Using the Theorems 2.1 and 2.4 matrices $P_3, P_4, P_5, P_6, P_7, P_{10}, P_{12}, P_{13}, P_{15}, P_{16}$ and P_{18} can be a parameter matrices. By using Theorem 2.3 matrices P_5, P_6, P_7 and P_{13} cannot be a parameter matrices, because of the number of white colors is not integer.

Consider the mapping T_1 and T_2 as follows:

$$T_1(a_1) = T_1(a_5) = T_1(a_{11}) = T_1(a_{15}) = W, \qquad T_1(a_2) = T_1(a_6) = T_1(a_{12}) = T_(a_{16}) = B,$$
$$T_1(a_3) = T_1(a_4) = T_1(a_7) = T_1(a_8) = T_1(a_9) = T_1(a_{10}) = T_1(a_{13}) = T_1(a_{14}) = R.$$

$$T_2(a_2) = T_2(a_3) = T_2(a_6) = T_2(a_7) = W, \qquad T_2(a_9) = T_2(a_{12}) = T_2(a_{13}) = T_2(a_{16}) = B,$$
$$T_2(a_1) = T_2(a_4) = T_2(a_5) = T_2(a_8) = T_2(a_{10}) = T_2(a_{11}) = T_2(a_{14}) = T_2(a_{15}) = R.$$

It is clear that T_1 and T_2 are perfect 3-coloring with the matrices P_7 and P_{13} respectivehy. □

Theorem 3.3. *The graph* $GP(8, 2)$ *has no perefect 3-colorings.*

Proof. A parameter matrix corresponding to perfect 3-colorings of the graph $GP(8, 2)$ may be one of the matrices $P_1, \cdots, P_{18}$. By using Theorem 2.1 and Theorem 2.4, we can see that only the matrices P_3, P_4, P_5, P_6, P_{10}, P_{12}, P_{15} and P_{18} can be a parameter matrices. By using Theorem 2.3, matrices P_4, P_5, P_{10}, P_{12}, P_{15}, P_{18} cannot be a parameter matrices, because the number of white, black and red, are not an integer. For matrix P_6, each vertex with color white has three adjacent vertices with color red. Now have the following possibilities:

(1) $T(a_1) = T(a_5) = T(a_7) = T(a_8) = T(a_{10}) = T(a_{11}) = T(a_{14}) = T(a_{15}) = R$, $T(a_2) = T(a_3) = B$, $T(a_6) = T(a_9) = T(a_{13}) = T(a_{16}) = W$ then $T(a_4) = B$ and $T(a_{12}) = W$, which is a contradiction with the second row of matrix P_6.

(2) $T(a_1) = T(a_5) = T(a_{11}) = T(a_{15}) = R$, $T(a_2) = T(a_9) = T(a_{12}) = T(a_{13}) = W$, $T(a_7) = T(a_8) = T(a_{14}) = T(a_{16}) = B$ then $T(a_3) = T(a_4) = R$, which is a contradiction with the three row of matrix P_6. Hence graph $GP(8, 2)$ has no perfect 3-colorings with the matrix P_6.

□

Theorem 3.4. *The graph* $GP(8, 3)$ *has no perefect 3-colorings.*

Proof. A parameter matrix corresponding to perfect 3-colorings of the graph GP(8, 3) may be one of the matrices $P_1, \cdots, P_{18}$. By using Theorem 2.1 and Theorem 2.4, we can see that only the matrices P_3, P_4, P_5, P_6, P_7, P_{10}, P_{12}, P_{13}, P_{15} and P_{18} can be a parameter matrices. By using Theorem 2.3, matrices P_3, P_4, P_5, P_{10}, P_{12}, P_{14}, P_{15}, and P_{18} cannot be a parameter matrices, because the number of white, black and red, are not an integer. For matrix P_6, each vertex with color white has three adjacent vertices with color red. Now have the following possibilities:

(1) $T(a_1) = T(a_3) = T(a_9) = T(a_{13}) = T(a_{14}) = R$, $T(a_4) = T(a_{10}) = T(a_{11}) = T(a_{15}) = T(a_{16}) = B$, $T(a_2) = T(a_8) = T(a_{12}) = W$ then $T(a_5) = B$ and $T(a_6) = T(a_7) = R$, which is a contradiction with the three row of matrix P_6.

(2) $T(a_1) = T(a_{11}) = T(a_{13}) = W$, $T(a_3) = T(a_4) = T(a_7) = T(a_8) = T(a_{15}) = B$, $T(a_2) = T(a_5) = T(a_6) = T(a_{10}) = T(a_{12}) = T(a_{16}) = R$ then $T(a_9) = T(a_{14}) = R$, which is a contradiction with the three row of matrix P_6. Hence graph $GP(8, 3)$ has no prtfrct 3-colorings with the matrix P_6.

Similar to matrix P_6,we can proof for the matrix P_7 as follows:

For matrix P_7 each vertex with color white has two adjacent vertices with color red. Now have the following two possibilities:

(3) $T(a_1) = T(a_4) = T(a_5) = T(a_8) = T(a_9) = T(a_{16}) = R$, $T(a_3) = T(a_6) = T(a_{10}) = T(a_{15}) = B$, $T(a_2) = T(a_{14}) = W$ then $T(a_7) = T(a_{11}) = W$ and $T(a_{12}) = T(a_{13}) = R$, which is a contradiction with the three row of matrix P_7.

(4) $T(a_1) = T(a_4) = T(a_9) = T(a_{12}) = W$, $T(a_5) = T(a_8) = T(a_{13}) = B$, $T(a_2) = T(a_3) = T(a_6) = T(a_7) = T(a_{10}) = T(a_{11}) = T(a_{14}) = R$ then $T(a_{15}) = T(a_{16}) = B$, which is a contradiction with the two row of matrix P_7. Hence graph $GP(8, 3)$ has no perfect 3-colorings with matrix P_7.

□

Finally, we summarize the results of this paper in the following table.

Table 1: Parameter matrices of some generalized peterson graph

Graphs	Parameter Matrices
graph GP(7,1)	×
graph GP(8,1)	P_7, P_{13}
graph GP(8,2)	×
graph GP(8,3)	×

References

[1] M. Alaeiyan and A. Mehrabani, Perfect 3-colorings of cubic graphs of order 10 , Electronic Journal of Graph Theory and Applications (EJGTA),5(2) (2017),PP.194-206. 2.2, 2.4

[2] M. Alaiyan, A. Mehrabani, Perfect 3- colorings of the platonic graph, Iranian J. Sci. Technol., Trans. A Sci. (2017) 1-9. 2.3

[3] S.V. Avgustinovich and I. Yu. Mogilnykh, Perfect 2-colorings of Johnson graphs J(6, 3) and $J(7,3)$, *Lecture Notes in Computer Science* 5228 (2008), 11-19. 1

[4] S.V. Avgustinovich and I. Yu. Mogilnykh, Perfect colorings of the Johnson graphs $J(8,3)$ and $J(8,4)$ with two colors, *Journal of Applied and Industrial Mathematics* 5 (2011), 19-30. 1

[5] D.G. Fon-Der-Flaass, A bound on correlation immunity, *Siberian Electronic Mathematical Reports Journal* 4 (2007), 133-135. 1

[6] D.G. Fon-Der-Flaass, Perfect 2-colorings of a hypercube, *Siberian Mathematical Journal* 4(2007), 923-930. 1

[7] D.G. Fon-der-Flaass, Perfect 2-colorings of a 12-dimensional Cube that achieve a bound of correlation immunity, *Siberian Mathematical Journal* 4 (2007), 292-295. 1

[8] A.L. Gavrilyuk and S.V. Goryainov, On perfect 2-colorings of Johnson graphs J(v, 3), *Journal of Combinatorial Designs* 21 (2013), 232-252. 1

[9] C. Godsil, Compact graphs and equitable partitions, *Linear Algebra and Its Application* 255(1997), 259-266. 1, 2.1